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A quasi-crisis in a quasi-dissipative system

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Abstract. A system concatenated by two area-preserving maps may be addressed as "quasi-dissipative", since such a system can display dissipative behaviors. This is due to noninvertibility induced by discontinuity in the system function. In such a system, the image set of the discontinuous border forms a chaotic quasi-attractor. At a critical control parameter value the quasi-attractor suddenly vanishes. The chaotic iterations escape, *via* a leaking hole, to an emergent period-8 elliptic island. The hole is the intersection of the chaotic quasi-attractor and the period-8 island. The chaotic quasi-attractor thus changes to chaotic quasi-transients. The scaling behavior that drives the quasi-crisis has been investigated numerically.

PACS. 05.45.Ac Low-dimensional chaos

1 Introduction

A crisis, which denotes sudden, discontinuous changes of chaotic attractors in conventional dissipative systems, was predicted and investigated by Grebogi, Ott, and Yorke [1–3]. They have mentioned mainly three types of crises. They are: interior crisis, boundary crisis, and cyclic crisis [2]. In the case of a boundary crisis, a chaotic attractor collides with an unstable periodic orbit on its basin boundary. After the crisis the boundary cuts out nearly all the points of the unstable manifold of the periodic orbit, which has the shape of the original chaotic attractor. The remnants form a fractal set that is addressed as a chaotic saddle [2,4,5]. Trajectories starting from points of a chaotic saddle never leave the saddle and exhibit chaotic motion forever. It is, however, completely unlikely to hit such a point by random choice since the saddle is a set of zero measure and is globally not attractive. What is observable experimentally is not the saddle but rather a small neighborhood of it. Trajectories starting close to the saddle can stay for a long time in its neighborhood and show chaotic properties, but sooner or later they escape. Thus the chaotic saddle leads to transient chaos. In some systems the chaotic transients may be superlong, and can be addressed as supertransients [6]. Properties of chaotic saddles and chaotic transients are important physical quantities in many practical fields, for example, in controlling chaos [7] and sustaining chaos [8].

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The most important property of crisis, the scaling law of the so-called "characteristic time" (or lifetime) τ, usually shows a universal form [2]:

$$\langle\tau\rangle \sim \epsilon^{-\nu}, \qquad \text{when} \quad \epsilon \to 0, \tag{1}$$

where $\langle\tau\rangle$ denotes the average value of τ and $\epsilon = |p - p_c|$ (p is the driving parameter and p_c is its critical value). The exact definition of τ and the method for obtaining its average value are different for different kinds of crises [2]. In everywhere smooth one-dimensional maps the scaling exponent ν takes a universal value 1/2. In an everywhere smooth two-dimensional map ν can be expressed as a function of the eigenvalues associated with the unstable periodic orbit responsible for the crisis [2].

This article presents a new kind of crisis. At the criticality point a so-called "chaotic quasi-attractor" suddenly vanishes, but the mechanism is qualitatively different from what was shown in boundary crisis. A supertransients appears after it. We cannot find any chaotic saddle, but can define a "strange repeller" [5] instead. This new kind of crisis was observed in so-called "quasi-dissipative systems" or "piece-wise smooth concatenations of two conservative systems", which basically do not obey the famous Kolmogorov-Arnold-Moser (KAM) theorem. In recent years some scientists have paid attention to such non-KAM systems [9–16]. Among them, references [13,14] discussed a system exemplified by a particle in an infinite potential well subject to a periodic kicking force. They found a kind of diffusion in a stochastic web structure with special scaling properties. Reference [15] reported an investigation in a system concatenated by two area-preserving maps, which can be viewed as the model of an electronic relaxation oscillator with over-voltage protection. Reference [16] studied a simplified model of the system discussed in reference [15]. The authors found that the systems can display dissipative behaviors due to non-invertibility induced by discontinuity of the system function. For example, some elliptic islands may attract iterations from outside due to the fact that there may be one of the backward images of a point inside the islands. As soon as the iterations enter the islands, they follow the conservative regulations exactly. The islands therefore were addressed as "regular quasi-attractors" [15,16], and the systems may be addressed as quasi-dissipative systems. We shall still use these two systems to show the new crisis.

The article is arranged as follows: Section 2 introduces two systems and some research results obtained with them; Section 3 reports the quasi-crisis; Section 4, the last section, contains a discussion and conclusion.

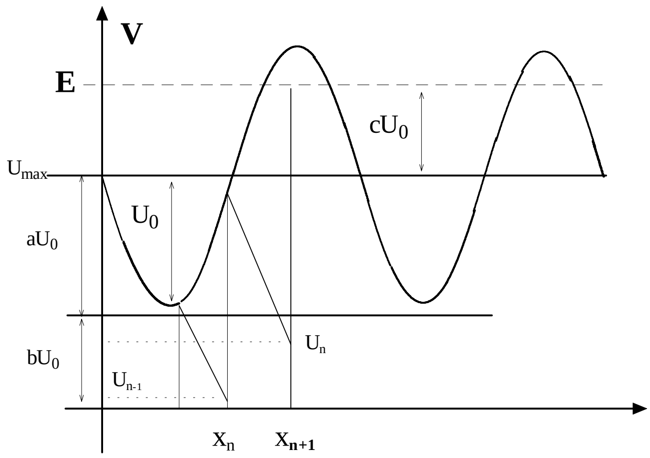

Fig. 1. A drawing showing the relaxation oscillation of the voltage across the capacitor and both the upper and lower thresholds, as well as the over-voltage protection.

2 The systems

Wang *et al.* suggested a model of an electronic relaxation oscillator with over-voltage protection [15]. In the circuit a capacitor is repeatedly charged and discharged, operated by two electronically controllable switches. The charging current is much larger than the discharging current. The voltage across the capacitor, V, varies linearly between a sine-modulated upper threshold and a lower threshold. For simplicity the time period of the charging duration is ignored. When considering over-voltage protection, we let the upper threshold equal a constant E in the phase region F where $V > E$. Thus the upper threshold can be expressed by two different forms

$$\begin{cases} U_n^{\mathrm{up}} = U_{\max} - U_0 \sin(2\pi x_n), & \text{when} \quad x_n \notin F, \\ U_n^{\mathrm{up}} = U_{\max} + c\,U_0, & \phantom{\text{when}} \quad x_n \in F, \end{cases} \tag{2}$$

where U_0 is the amplitude of the upper threshold modulation signal, and $U_{\max}$ and c are constants, x_n is the normalized phase of the modulation signal (at the point of the upper threshold where the relaxation oscillation reaches at time t_n), $F = (x_{F_1}, x_{F_2}) = (0.5 + (\arcsin c)/2\pi, 1 - (\arcsin c)/2\pi)$ (this being the phase region of over-voltage protection). The lower threshold is modulated by the underlying x_n phase according to a certain rule. We define another variable, y_n, to describe it. Figure 1 shows the relaxation oscillation of V and both the upper and lower thresholds, as well as the over-voltage protection. Based on the characteristics of the circuit one can get the following map describing the system [15]:

$$\begin{cases} x_{n+1} = f_{1x} = x_n + y_{n+1} + \frac{a}{b}, \\ y_{n+1} = f_{1y} = y_n - \frac{\sin 2\pi x_n}{b}, \end{cases} \text{[mod. 1] when } x_n \notin F, \tag{3}$$

$$\begin{cases} x_{n+1} = f_{2x} = x_n + y_n + \frac{a+c}{b}, \\ y_{n+1} = f_{2y} = y_n + 2x_n, \end{cases} \text{[mod. 1] when } x_n \in F, \tag{4}$$

where a, b are constant parameters. If there is no voltage protection, the linear map (4) does not appear. The remaining map (3) can be viewed as a kind of standard map [17]. The main characteristics of it have been discussed already in many references. The authors of reference [15] show that in map (3) the last remaining KAM torus which stretches from $x = 0$ to $x = 1.0$ is going to be broken at $b = b_c = 6.46684...$ When $b < b_c$, the system is globally chaotic. We denote the two borderlines between the definition ranges of map (3) and (4) by $\{(x, y)|x = x_{F_1}\}$ and $\{(x, y)|x = x_{F_2}\}$. When voltage protection is applied they may hit some KAM cycles in an elliptic island and destroy them.

One should note that there might be some energy lost when over-voltage protection is applied. Thus this system can be dissipative at this point and conservative in other places in the process. While the system changes between dissipative and conservative, the absolute value of the determinant of the Jacobean matrix of maps (3) or (4) equals a unit. The system can show a kind of non-invertibility that leads to dissipative behaviors. The inverse mappings of maps (3) and (4) are:

$$\begin{cases} x_n = f_{1x}^{-1} = x_{n+1} - y_{n+1} - a/b, \\ y_n = f_{1y}^{-1} = y_{n+1} + \sin(2\pi x_n)/b, \end{cases} \text{when } x_n \notin F, \tag{5}$$

$$\begin{cases} x_n = f_{2x}^{-1} = -x_{n+1} + y_{n+1} + (a+c)/b, \\ y_n = f_{2y}^{-1} = y_{n+1} - 2x_n, \end{cases} \text{when } x_n \in F. \tag{6}$$

Due to the fact that the condition of selecting solutions in mappings (5) or (6) is determined by x_n instead of x_{n+1}, one can find two (x_n, y_n) values for each (x_{n+1}, y_{n+1}) according to either the function f_1 or f_2. That means the concatenation map is noninvertible.

The authors of reference [15] discussed the evolution of three fixed points. They are at $(x_1^*, y_1^*) = (0, -a/b)$,

$(x_2^*, y_2^*) = ((\arcsin b)/2\pi, -a/b)$ and $(x_3^*, y_3^*) = (1 - (\arcsin b)/2\pi, -a/b)$. These fixed points are elliptic when the condition $b > \pi/2$ (for (x_1^*, y_1^*)), or $b \in (\pi/\sqrt{4+\pi^2}, 1)$ (for $(x_{2,3}^*, y_{2,3}^*)$) is satisfied, respectively. With the parameter values $a = 2.0$ and $c = 0.933564$, the authors of reference [15] show analytically and numerically that (x_1^*, y_1^*) period doubles first at $b = \pi/2 \simeq 1.570$. The period-doubling bifurcation cascade is normal before $b \simeq 1.37$ when one of the period-2 elliptic islands hits a borderline. Between this value and $b \simeq 1.324$ more and more KAM cycles of the island vanish *via* collisions with the border. This entire process happens beneath the threshold b_c, *i.e.*, inside the globally chaotic region. If there were no voltage protection the islands would have been inside a chaotic sea. But in the current system all the chaotic trajectories escape, after chaotic transients, to the islands *via* one or two leaking holes. A hole is mainly formed by the intersection of the inverse image of one of the islands and the protection region F. As soon as the iterations reach the islands, they perform typical conservative properties. The islands thus were addressed as "regular quasi-attractors" and the chaotic transients were addressed as "chaotic quasi-transients". All the chaotic quasi-transients become stable chaotic orbits at $b \simeq 1.324$ due to the vanishing of the leaking hole, *i.e.*, due to the fact that the inverse image of the island leaves the protection region, but there is still a remaining part of the period-2 elliptic island. The main period doubling bifurcation cascade is interrupted by a hitting of the period-2 elliptic point to the borderline at $b \simeq 1.21774$, so that the system shows a complete chaotic motion. After $b = 1.0$ a new elliptic point (x_3^*, y_3^*) appears, and the trajectories in the chaotic sea change to chaotic quasi-transients again by similar reason. The aforementioned elliptic point (x_2^*, y_2^*) falls into the protection region and does not appear. With $a = 2.0$, $b = 0.933564 < 1$, and parameter c varying inside the range $c \in [0.7, 0.9]$, the authors of reference [15] show all the quasi-dissipative properties more clearly.

As is well-known, the behaviors of an elliptic island surrounded by a chaotic sea in the phase space of a standard map can be described by a DeVogelaere square mapping [17,18]. Therefore all the above-mentioned behaviors can be qualitatively displayed by a piece-wise smooth concatenation of a DeVogelaere square mapping and a linear map. Wang *et al.* [16] used such a simplified model and made more analytic discussion on quasi-dissipative behaviors. Although the two systems have their own characteristics the quasi-dissipative behaviors they show are qualitatively same. So we may think that this simplified model has a similar electronics background as maps (3) and (4). We shall mainly discuss this model [16] in the current paper. The simplified maps read:

$$\begin{cases} x_{n+1} = g_{x1} = px_n - (1-p)x_n^2 - y_n, \\ y_{n+1} = g_{y1} = x_n - px_{n+1} + (1-p)x_{n+1}^2, \end{cases} \text{when } x_n \geq f, \tag{7}$$

$$\begin{cases} x_{n+1} = g_{x2} = x_n + c, \\ y_{n+1} = g_{y2} = y_n + c, \end{cases} \text{when} \quad x_n < f, \tag{8}$$

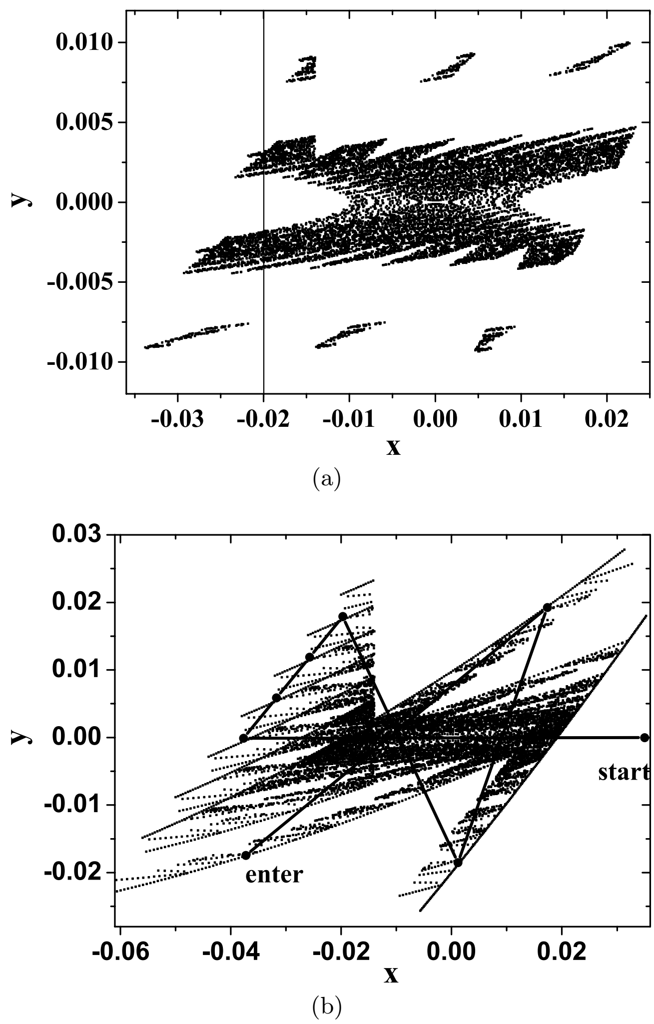

Fig. 2. This figure shows a chaotic quasi-attractor (a) and the quasi-transience, as well as the set of images of the discontinuous border observed in systems (7) and (8) (b). The parameter values and the computation methods are indicated in the text.

where p, c, and f are real constants. There is only one borderline, $\{(x, y)|x = x_f\}$, between the definition ranges of map (7) and (8). In a square mapping, iterations from most of the initial points outside the stable islands trend to infinity. We ignore these iterations in the current discussion. It is easy to verify that maps (7) and (8) have similar noninvertible and quasi-dissipative properties [16].

3 The quasi-crisis

3.1 The chaotic quasi-attractor

In the simplified models (7) and (8), a chaotic quasi-attractor can be formed by the set of discontinuous border images. To show this phenomenon, Figure 2a was computed by using (7) and (8) and selecting an initial value $(-0.035, -0.0075)$ in phase space. The parameter values were chosen as $p = -1.007$, $c = 0.006$, and $f = -0.02$.

We ignored the first 1 000 iterations to avoid the quasi-transients, and then recorded the following 20 000. When we record 20 000 iterations from the initial value without ignoring the first 1 000 ones, the obtained pattern occupies a much larger part of the phase space. Thus Figure 2a should show a chaotic quasi-attractor. Figure 2b is designed to show the quasi-transience and the set of images of the discontinuous border with the same parameter values. The black circles indicate the first 7 iterations from an initial value $(0.035, 0.0)$, the lines and the small spots indicate the computation results obtained by recording 450 iterations from 1 000 evenly distributed initial values on the discontinuous borderline $\{(x, y)|_{x=f;y\in[-0.01592,0.02521]}\}$. Here we confine y in the range $y \in [-0.01592, 0.02521]$ because the iterations from most of the initial values beyond this range will trend to infinity. When we ignore about 300 iterations from the border, the remaining iterations form exactly the same pattern as shown in Figure 2a and as shown by the dense black spots in Figure 2b. Thus one can clearly see that the transient iterations from $(0.035, 0.0)$ tend to go toward and then fall into the chaotic quasi-attractor, which is formed by the end-results of the border images. A similar phenomenon was also observed in the original systems (3) and (4). Actually, the "stable chaotic orbits" or "complete chaotic motion", those appearing between $b \simeq 1.324$ and $b \simeq 1.21774$ or $b \simeq 1.21774$ and $b = 1.0$, and are mentioned at the end of Section 2, can be certified as chaotic quasi-attractors. They will even look like typical chaotic attractors in dissipative systems if parameter c becomes smaller and the protection region becomes larger. This conclusion was not reported in reference [15]. The details will be presented elsewhere.

3.2 The quasi-crisis

The chaotic quasi-attractor in maps (7) and (8) suddenly disappears and the so-called "quasi-crisis" happens at $p = p_c \simeq -1.00697995$. The crisis is addressed as such because it describes a sudden change in a chaotic quasi-attractor. At the criticality point, a period-8 elliptic orbit suddenly emerges inside the chaotic quasi-attractor. One of the elliptic points is adjacent to the discontinuous borderline. When p becomes smaller still, the elliptic islands around the elliptic orbit become continuously larger. However, the largest KAM cycle in the island nearest to the borderline is always tangent to the line. The iterations from all the initial points (which do not tend to infinity) are attracted to the island as shown by Figure 3. The original chaotic quasi-attractor changes to chaotic transients. Like what is shown in all the conventional crises [1,2,6], the chaotic transience is extremely long. Usually it takes 10^6, sometimes even 10^8, iterations. Figure 3a shows part of the chaotic quasi-transients and the regular quasi-attractor together. The latter is too small to be seen in Figure 3a. To solve this problem, Figure 3b shows a magnification of the part of the phase space shown by the square in Figure 3a, where one of the 8 elliptic islands intersects with the set of discontinuous border images shown by small spots. The chaotic transient iterations now escape from the intersection set that serves as 8 leaking holes. We note that very recently Buljan and Paar investigated the escape of iterations in a chaotic attractor from many holes [19]. They proved analytically that the

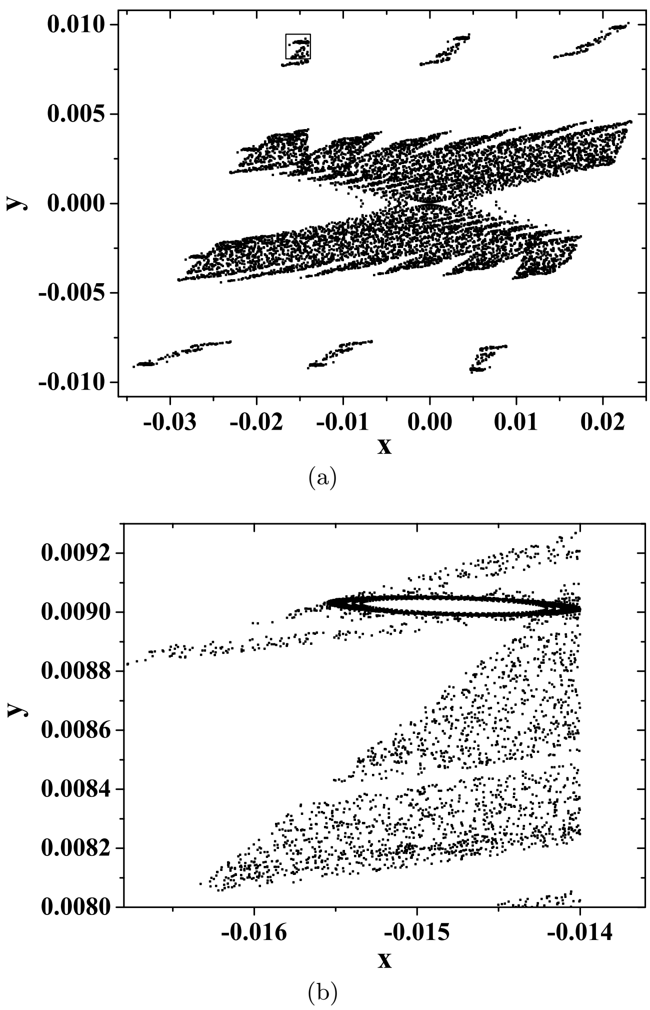

Fig. 3. (a) This figure shows part of the chaotic quasi-transients and the period-8 elliptic island chain. The computation was conducted by maps (7) and (8) and the parameter values $p = -1.006$, $f = -0.02$, and $c = 0.006$. The first 10^4 iterations from the 8×8 initial values, which evenly distributed in the phase space area $x \in [-0.015, 0.01]$, $y \in [-0.006, 0.006]$, were ignored, and then the following 2 500 iterations were recorded. When we ignore 10^9 iterations from the initial values, the following iterations fall in the period-8 elliptic islands. The largest KAM cycle in the island nearest to the borderline is always tangent to the line. (b) shows some KAM circles in another one of the islands. All the iterations showing the chaotic transient pattern in (a) disappear at that point. (b) This figure shows a magnification of the part of the phase space shown by the square in (a). The small spots show the discontinuous border images. The larger circles show some of the KAM circles in one of the period-8 elliptic islands, which surrounds the corresponding elliptic point.

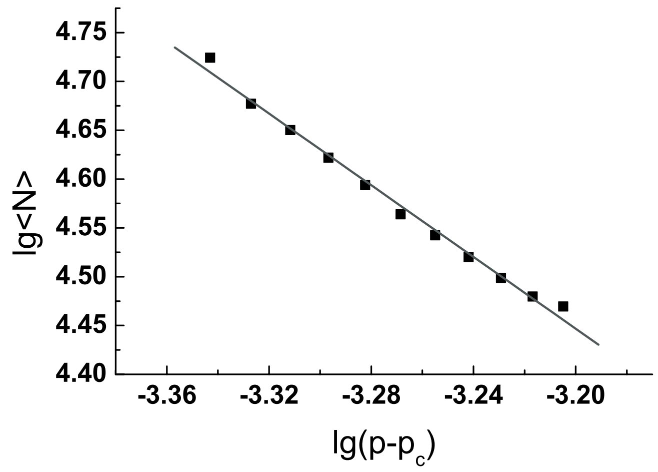

Fig. 4. The black squares show the numerical data computed by maps (7) and (8), and the definition (10). The parameter values are chosen as $c = 0.006$, $f = -0.02$, and $p_c = -1.0069799$. Parameter p varies in the range $p \in [-1.0067, -1.0063]$. The range is small due to the fact that when $p > -1.0063$ the elliptic orbits, which intersect with the image set of the border, show sudden changes. In the computation, $\langle N \rangle$ was obtained by calculating the iterations from evenly distributed 201×201 initial values. The good agreement of the data to the linear line, which was obtained by the least square fitting, indicates that this scaling law (9) is valid in the underlying system.

many-hole interactions can significantly prolong the average lifetime. This is in agreement with our results about the superlong chaotic transients.

This behavior resembles an escape from a strange set [20]. We thus can expect a scaling behavior as:

$$\langle N \rangle \propto (p - p_c)^{-\nu}, \qquad \text{when} \quad p \to p_c, \tag{9}$$

where the mean transient time $\langle N \rangle$ is defined as:

$$\langle N \rangle = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} N_i}{n}. \tag{10}$$

In the definition n is the number of initial points, from which the iterations tend toward the regular quasi-attractor, and N_i is the length of quasi-transients from each initial point. This scaling law is in qualitative agreement with equation (1). The fact also suggests that the current phenomenon belongs to the crises. As shown by Figure 4, our numerical investigation certifies that the scaling law (9) is valid in the underlying system, and that the scaling exponent $\nu = 1.66 \pm 0.04$ when the critical parameter value equals $p_c = -1.0069799$.

3.3 Similar behaviors in other systems

Similar phenomena with qualitatively the same scaling behavior were also discovered in other quasi-dissipative systems. In the original model of the electronic relaxation oscillator, (3) and (4), as briefly introduced in Section 2, the authors of reference [15] found a period-2 elliptic orbit coming from the period doubling bifurcation of the fixed point (x_1^*, y_1^*). The orbit hits the borderline at $b \simeq 1.21774$, so that the system shows a complete chaotic motion. We certified that the chaotic iterations form a chaotic quasi-attractor. As very briefly mentioned in reference [15], the similar scaling behavior for the mean transient time can be found numerically. The obtained rule is $\langle N \rangle \propto (b - b_c)^{-\nu}$. Here the scaling exponent is $\nu \simeq 1.39$. However, the critical value is $b_c = 1.3245$ instead of $b \simeq 1.21774$. The reason is that the escaping hole vanishes, before the collision of the period-2 elliptic orbit with the discontinuous borderline, due to the fact that the inverse image of one of the elliptic islands leaves the protection region F at $b_k = 1.3245$. This fact makes the behavior differ from a quasi-crisis, but it still resembles an escape from a strange set.

Another system that displays similar behavior is a model describing the motion of a kicked particle. As stated in Section 1, references [13,14] discussed a system exemplified by a particle in an infinite potential well along half of a cycle subject to a periodic kicking force. This system can be described by a two-dimensional map that is discontinuous but invertible [13,14]. The system thus is fully conservative. We studied a revised version of the model in which the particle still moves along the cycle. In the upper or lower half it moves in the same potential but is subjected to a kicking force with different periods. The revised system can be described by a two-dimensional map that is discontinuous and noninvertible, which means it becomes quasi-dissipative. When the ratio between the periods of the kicking force, β, decreases from 1, the new system displays a transition from conservative to quasi-dissipative. We found that, after the criticality point $\beta_c = 1$, many elliptic islands intersect with a stochastic web, along which the iterations had performed fully conservative chaotic diffusion. The intersection set between the islands and the web then serve as many leaking holes, and the chaotic diffusion becomes superlong chaotic transients. This phenomenon also differs from the quasi-crisis, but it also resembles an escape from a strange set. The numerically computed scaling rule is $\langle N \rangle \propto (\beta - \beta_c)^{-\nu}$. Here the scaling exponent is $\nu \simeq 1.148$.

The details of the phenomena briefly reported in this subsection will be presented elsewhere.

4 Discussion and conclusion

In conclusion we have observed a new kind of crisis in a quasi-dissipative system. The crisis is induced by a mechanism different from those in conventional types. The new mechanism is the emergence of a periodic elliptic island chain inside a chaotic quasi-attractor. This new crisis obeys a scaling law with a similar form to what is shown in conventional crises. The scaling exponent is in a reasonable range [20], but we cannot understand it in an analytical way yet. Another difference between the quasi-crisis and conventional types is that we cannot find any chaotic saddle. We believe it does not exist since the mechanism, which produces chaotic saddles as introduced in Section 1, does not exist here. However, by ensemble method [5] we can find points of the iterations, which stay in the vicinity of an invariant set. Trajectories starting from points of the set never leave it and exhibit chaotic motion forever. This fact means that the inverse image set of the period-8 elliptic islands cut out nearly all the points of the end-results of the border images. The remnants form a fractal set. Following Tél, we can define this set as a "strange repeller" [5].

The quasi-crisis is probably common in these types of systems. If one investigates the crisis along the opposite direction of the control parameter variation, the phenomenon becomes a quasi-intermittency. This is induced by a collision between a periodic elliptic orbit and the discontinuous border of the system function. However, the scaling law is very different from that of the conventional intermittency.

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